then there exist integers r, s *such that*

$$b = rq_n + sq_{n-1}, \qquad a = rp_n + sp_{n-1}$$

and $|s| \leqslant A+1$. *In addition* $(a, b) = (r, s)$ *and*

$$|b\xi - a| = \frac{rq_n + sq_{n-1} - sq'_{n+1}}{q_n q'_{n+1}}.$$

If we put

$$M_2 = \min_b \min_{(a,b)=1} b\,|b\xi - a|,$$

where the first min is only over b satisfying

$$N \leqslant b \leqslant \min(q_{m+1}, cN),$$

then this lemma allows us to express the condition $M_2 \leqslant A$ again as a condition on the variables in (3.1). In fact, since s in lemma 4 is limited to finitely many values, one can write M_2 as a minimum of finitely many simple expressions in these variables in the region $M_2 \leqslant A$.

Since

$$S(N, A, c) = \{\xi\colon M_1 \leqslant A \text{ or } M_2 \leqslant A\},$$

one can then conclude that $\lim |S(N, A, c)|$ exist from the existence of the limiting distribution in lemma 3.

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On a conjecture of Erdös and Szüsz related to uniform distribution mod 1

by

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1. Introduction. Let $\xi \in [0, 1]$, $0 \leqslant a < b \leqslant 1$, and denote by $N(M, \xi, a, b)$ the number of integers k, $1 \leqslant k \leqslant M$, for which $a \leqslant \{k\xi\} < b$. ($\{c\}$ denotes the fractional part of c). Our main result gives a criterion for the boundedness of

(1.1) $$R(M, \xi, a, b) = N(M, \xi, a, b) - M(b-a).$$

This is stated in

THEOREM 4. *For* $0 \leqslant a < b \leqslant 1$, $b-a < 1$ *and fixed* ξ, $R(M, \xi, a, b)$ *is bounded in* M *if and only if*

(1.2) $$b - a = \{j\xi\} \quad \text{for some integer } j.$$

It was known for a long time (cf. [6], [10]) that (1.2) is a sufficient condition for the boundedness of R and the result that (1.2) is also necessary confirms a recent conjecture of Erdös and Szüsz [2].

Throughout this paper we shall make heavy use of continued fraction expansions in the following notations:

The regular continued fraction of an irrational(1)(2) $\xi \in (0, 1)$ is denoted by

$$[a_1(\xi), a_2(\xi), \ldots] = \cfrac{1}{a_1(\xi) + \cfrac{1}{a_2(\xi) \ldots}}$$

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(1) We shall ignore rational ξ's most of the time. They form a set of measure zero and therefore do not influence the metric result in section 3. Also they constitute a trivial case for theorem 4.

(2) We use the notation of Chapter 10 of [5] except that we drop $a_0(\xi) = [\xi]$ from our formulae, since $a_0(\xi) = 0$ in all our considerations.

and its nth convergent by $p_n(\xi)/q_n(\xi)$. One has then the well-known recursion formulae ([5], chapter 10)

$$q_0 = 1, \quad q_1 = a_1, \quad q_{n+1} = a_{n+1}q_n + q_{n-1}, \tag{1.3}$$

$$p_0 = 0, \quad p_1 = 1, \quad p_{n+1} = a_{n+1}p_n + p_{n-1}. \tag{1.4}$$

We introduce also

$$a'_{n+1} = a'_{n+1}(\xi) = a_{n+1} + [a_{n+2}, a_{n+3}, \ldots] = a_{n+1} + \cfrac{1}{a_{n+2} + \cfrac{1}{a_{n+3} \ldots}} = a_{n+1} + \frac{1}{a'_{n+2}} \tag{1.5}$$

and

$$q'_{n+1} = q'_{n+1}(\xi) = a'_{n+1}q_n + q_{n-1} = q_{n+1} + \frac{q_n}{a'_{n+2}} = \frac{q'_{n+2}}{a'_{n+2}}. \tag{1.6}$$

As in Ostrowski [9], one can expand N as

$$N = \sum c_i q_i = \sum_{i=0}^{m(N,\xi)} c_i(N, \xi) q_i(\xi) \tag{1.7}$$

where

$$0 \leqslant c_i \leqslant a_{i+1}, \quad c_{m(N,\xi)} > 0, \quad \text{and} \quad \sum_{i=0}^{j} c_i q_i < q_{j+1}$$

$$\text{for} \quad 0 \leqslant j \leqslant m(N, \xi).$$

Such an expansion exists and is uniquely determined by these conditions (see [9] and [11, part I, p. 464]). The letter m will be reserved for the (finite) upper bound $m(N, \xi)$ in (1.7). When no confusion is likely we do not write the arguments N and ξ. Note that q_m is the last denominator of a convergent of ξ, which does not exceed N.

To prove theorem 4 we begin with a detailed study of the N intervals into which $[0,1]$ is divided by the points $\{k\xi\}$, $k = 1, \ldots, N$. We shall always identify the points 0 and 1 and accordingly consider $[\max\{k\xi\}, 1] \cup [0, \min\{k\xi\}]$ as one interval so that the N points divide $[0,1]$ into N rather than $N+1$ subintervals. The lengths and relative location of these subintervals are described by theorem 1 and corollary 1 in terms of the quantities c_i, q_m, q'_{m+1} and a'_{m+1}. Corollary 1 once more confirms a conjecture of Steinhaus that, for each N, subintervals of only three different lengths occur. This conjecture was proved before by Surányi [13], by means of the Farey series F_N. F_N is the sequence of rational numbers j/k, with $0 \leqslant j \leqslant k \leqslant N$ and $(j, k) = 1$, arranged in ascending order.

When Surányi's result is combined with theorem 1, one obtains the following amusing result:

THEOREM 2. *Let ξ be an irrational number such that $j_1/k_1 < \xi < j_2/k_2$ where j_1/k_1 and j_2/k_2 are successive members of F_N. Then, for*

$$\frac{j_1}{k_1} < \xi < \frac{j_1+j_2}{k_1+k_2}, \qquad m = m(N, \xi) \text{ is even},$$

$$q_m = q_{m(N,\xi)}(\xi) = k_1, \tag{1.8}$$

$$q_{m-1} = k_2 - \left[\frac{k_2}{k_1}\right]k_1 \quad (\text{here we define } q_{-1} = 0), \tag{1.9}$$

and

$$\xi - \frac{j_1}{k_1} = \frac{1}{q_m q'_{m+1}}. \tag{1.10}$$

If $\dfrac{j_1+j_2}{k_1+k_2} < \xi < \dfrac{j_2}{k_2}$, *then m is odd,*

$$q_m = k_2, \tag{1.11}$$

$$q_{m-1} = k_1 - \left[\frac{k_1}{k_2}\right]k_2, \quad \text{provided } k_2 > 1, \tag{1.12}$$

and

$$\frac{j_2}{k_2} - \xi = \frac{1}{q_m q'_{m+1}}. \tag{1.13}$$

As a byproduct of these results we derive a metric result concerning the maximal spacing between the points $\{k\xi\}$.

THEOREM 3. *Let*

$$L_N(\xi) = \max(\{k_2\xi\} - \{k_1\xi\})$$

where the maximum is over all pairs k_1, k_2 with $1 \leqslant k_1, k_2 \leqslant N$, $\{k_1\xi\} < \{k_2\xi\}$ such that there is no $1 \leqslant k_3 \leqslant N$ with $\{k_1\xi\} < \{k_3\xi\} < \{k_2\xi\}$. Consistent with the identification of 0 and 1 we also include the pair

$$\{k_1\xi\} = \max_{1 \leqslant k \leqslant N} \{k\xi\}, \quad \{k_2\xi\} = \min_{1 \leqslant k \leqslant N} \{k\xi\}$$

in which case $\{k_2\xi\} - \{k_1\xi\}$ is to be replaced by $1 - \{k_1\xi\} + \{k_2\xi\}$. (Roughly speaking, L_N is the maximum distance between adjacent points $\{k\xi\}$, $1 \leqslant k \leqslant N$.)

Then(3),

$$\lim_{N\to\infty} |\{\xi : NL_N(\xi) \leqslant x\}| =$$

$$= \begin{cases} 0 & \text{if} \quad x < 1, \\ \dfrac{12}{\pi^2} \displaystyle\int_{x^{-1}}^{1} \frac{xt-1}{t} \log \frac{x(2t-1)}{xt-1} dt + \frac{12}{\pi^2} \int_{x^{-1}}^{1} \frac{\log xt}{t} dt & \text{if} \quad 1 \leqslant x \leqslant 2, \\ \dfrac{12}{\pi^2} \displaystyle\int_{1-x^{-1}}^{1} \frac{xt-1}{t} \log \frac{x(2t-1)}{xt-1} dt + \frac{12}{\pi^2} \int_{1/2}^{1-x^{-1}} \frac{1}{t} \log \frac{t}{1-t} dt + & \\ \qquad\qquad + \dfrac{12}{\pi^2} \displaystyle\int_{1-x^{-1}}^{1} \frac{\log xt}{t} dt & \text{if} \quad 2 < x. \end{cases}$$

Theorem 3 is proved by the methods of Friedman and Niven [4] and Erdös, Szüsz and Turán [3] who also used Farey series. The author has used those techniques elsewhere [7] to derive the limiting distribution(4)

$$\lim_{N\to\infty} |\{\xi : 0 \leqslant \xi \leqslant 1,\ N \min_{1 \leqslant k \leqslant N} \|k\xi - \alpha\| \leqslant x\}|.$$

in case $\alpha = 0$. It seems that the techniques of the present paper are strong enough to treat the case of general α but the computations become too complicated to be carried out.

2. The successive values of $\{k\xi\}$. A large part of the information in this section can be found in, or derived from, V. Sós [11] and [12]. It is more convenient though, to give direct derivations here, which are adapted to the needs in section 4. Throughout this section N and ξ will be fixed, ξ irrational. N will be expanded as in (1.7) and m will stand for $m(N, \xi)$. As before the points 0 and 1 will be identified. q_{-1} is defined as zero.

THEOREM 1. *Each interval* $\left(\frac{r}{q_m}, \frac{r+1}{q_m}\right)$, $r = 0, 1, \ldots, q_m - 1$ *contains exactly one point* $\{k\xi\}$ *with* $1 \leqslant k \leqslant q_m$. *Denote the point in* $\left(\frac{r}{q_m}, \frac{r+1}{q_m}\right)$ *by* P_r *and the interval* $[P_r, P_{r+1})$ *by* J_r *in case* m *is even. If* m *is odd, let* P_r *be the point in* $\left(\frac{q_m-r-1}{q_m}, \frac{q_m-r}{q_m}\right)$ *and* J_r *the interval* $(P_{r+1}, P_r]$. *Then exactly* $q_m - q_{m-1}$ *intervals* J_r *have length* $\frac{a'_{m+1}}{q'_{m+1}}$ *and exactly* q_{m-1} *have length* $\frac{a'_{m+1}+1}{q'_{m+1}}$. *Intervals of the first set are called "short" and intervals of the second set are called "long". The long intervals are exactly those* J_r *for which*(5) $P_r = \{k\xi\}$ *with* $1 \leqslant k \leqslant q_{m-1}$. *The next* $(c_m - 1)q_m$ *points* $\{k\xi\}$, $q_m + 1 \leqslant k \leqslant c_m q_m$, *subdivide the intervals* J_r *in such a manner that exactly* $(c_m - 1)$ *points fall in each* J_r, *namely at the points*

$$P_r + \frac{(-1)^m s}{q'_{m+1}}, \quad s = 1, 2, \ldots, c_m - 1, \ r = 0, 1, \ldots, q_m - 1.$$

These points divide each J_r *into* c_m *sub-intervals. Starting from* P_r *the first* $c_m - 1$ *subintervals of* J_r *have length* $\frac{1}{q'_{m+1}}$ *and the last interval, adjacent to* P_{r+1} *and to be denoted by* J'_r, *has length* $\frac{a'_{m+1} - c_m + 1}{q'_{m+1}}$ *if* J_r *is short and length* $\frac{a'_{m+1} - c_m + 2}{q'_{m+1}}$ *if* J_r *is long.* J'_r *is called short or long when* J_r *is short, respectively long. Of the last* $N - c_m q_m$ *points* $\{k\xi\}$, $c_m q_m + 1 \leqslant k \leqslant N$, *at most one will belong to each* J_r. *If such a point belongs to* J_r, *it is located at* $P_r + \frac{(-1)^m c_m}{q'_{m+1}}$. *Such a point therefore belongs to* J'_r *and divides* J'_r *into an interval of length* $\frac{1}{q'_{m+1}}$ *adjacent to the previous intervals of length* $\frac{1}{q'_{m+1}}$ *in* J_r (*or adjacent to* P_r *if* $c_m = 1$) *and an interval* J''_r, *adjacent to* P_{r+1}. *These last* $N - c_m q_m$ *points* $\{k\xi\}$ *subdivide as many long* J'_r *as possible. I.e. if* $N - c_m q_m \leqslant q_{m-1}$ = *number of long* J_r, *then these points fall only in long* J_r. *If* $N - c_m q_m > q_{m-1}$ *then one such point falls in each long* J_r *and some points fall in a short* J_r.

Proof. Only the case of even m will be considered, the case where m is odd being entirely analogous(6). By the well-known formula ([5], chapter 10)

$$\xi = \frac{p_j}{q_j} + \frac{(-1)^j}{q_j q'_{j+1}}, \tag{2.1}$$

(3) $|A|$ denotes the Lebesgue measure of the set A.

(4) $\|\beta\|$ denotes the distance between β and the nearest integer to β.

(5) We slightly abuse notation and confuse P_r with the value of its coordinate in $[0, 1]$. This will often be done in the sequel.

(6) Some special considerations are necessary when $m = 0$, which corresponds to the case $0 < \xi < (N+1)^{-1}$. However, it is easy to see that the theorem remains valid in this case if one takes $q'_1 = a_1$, in agreement with (1.6).

we have for even m and $1 \leqslant k \leqslant q_m$

$$\{k\xi\} = \left\{\frac{kp_m}{q_m} + \frac{k}{q_m q'_{m+1}}\right\} = \frac{\varrho_k}{q_m} + \frac{k}{q_m q'_{m+1}}. \tag{2.2}$$

where ϱ_k is defined by

$$kp_m = \varrho_k (\operatorname{mod} q_m) \quad \text{and} \quad 0 \leqslant \varrho_k \leqslant q_m - 1. \tag{2.3}$$

As k runs through the values $1, \ldots, q_m$, ϱ_k runs through the values $0, \ldots, q_m-1$ since $(p_m, q_m) = 1$. Moreover,

$$\{k\xi\} \in \left(\frac{\varrho_k}{q_m}, \frac{\varrho_k+1}{q_m}\right), \tag{2.4}$$

since $0 < k/q_m q'_{m+1} < 1/q_m$. This shows that for each $r = 0, \ldots, q_m-1$. exactly one point

$$\{k\xi\} \in \left(\frac{r}{q_m}, \frac{r+1}{q_m}\right) \quad \text{with} \quad k = 1, \ldots, q_m.$$

This point is called P_r and the length of(7) $J_r = [P_r, P_{r+1})$ is

$$\frac{1}{q_m} + \frac{\lambda_{r+1} - \lambda_r}{q_m q'_{m+1}}$$

if λ_r is defined by

$$P_r = \{\lambda_r \xi\} = \frac{r}{q_m} + \frac{\lambda_r}{q_m q'_{m+1}}. \tag{2.5}$$

This of course means that (for m even) λ_r is the solution of

$$\lambda_r p_m \equiv r (\operatorname{mod} q_m) \quad \text{and} \quad 1 \leqslant \lambda_r \leqslant q_m. \tag{2.6}$$

Consequently

$$(\lambda_{r+1} - \lambda_r) p_m \equiv 1 (\operatorname{mod} q_m).$$

When combined with the standard formula ([5] chapter 10)

$$p_m q_{m-1} - p_{m-1} q_m = (-1)^{m-1}, \tag{2.7}$$

this gives

$$\lambda_{r+1} - \lambda_r \equiv -q_{m-1} (\operatorname{mod} q_m).$$

In view of $1 \leqslant \lambda_r \leqslant q_m$ we finally conclude

$$\lambda_{r+1} - \lambda_r = \begin{cases} -q_{m-1} & \text{if} \quad q_{m-1} < \lambda_r \leqslant q_m, \\ q_m - q_{m-1} & \text{if} \quad 1 \leqslant \lambda_r \leqslant q_{m-1}. \end{cases} \tag{2.8}$$

(7) In case $j = q_m - 1$, P_{j+1} is identified with P_0.

In the corresponding cases one has

$$|P_{r+1} - P_r| = \begin{cases} \dfrac{q'_{m+1} - q_{m-1}}{q_m q'_{m+1}} = \dfrac{a'_{m+1}}{q'_{m+1}}, \\ \dfrac{a'_{m+1}+1}{q'_{m+1}}. \end{cases} \tag{2.9}$$

As stated in the theorem there are therefore $q_m - q_{m-1}$ "short" intervals and q_{m-1} "long" intervals, the latter occuring if $P_r = \{\lambda_r \xi\}$ with $1 \leqslant \lambda_r \leqslant q_{m-1}$. The remaining statements concerning the subdivision of J_r are immediate now since, by (2.2) and (2.5),

$$\{(\lambda_r + s q_m)\xi\} = P_r + \frac{s}{q'_{m+1}} \in \left(P_r, \frac{r+1}{q_m}\right) \subseteq J_r \tag{2.10}$$

as long as $\lambda_r + s q_m \leqslant q'_{m+1}$ and thus in particular for $\lambda_r + s q_m \leqslant N < q_{m+1}$. The only part not yet proved so far is the statement that the points $\{k\xi\}$ with $c_m q_m + 1 \leqslant k \leqslant N$ first subdivide the long intervals J'_r. This again follows from (2.8), (2.9), and (2.10). In fact, k will be of the form $c_m q_m + \lambda_r$ and $\{k\xi\} \in J_r$ for some r. The values of $k \leqslant c_m q_m + q_{m-1}$ correspond to $\lambda_r \leqslant q_{m-1}$ and thus to the long intervals. These values of k precede the ones corresponding to short intervals, namely those with $k > c_m q_m + q_{m-1}$.

Corollary 1. *Among the N intervals into which $[0, 1]$ is divided by the points $\{k\xi\}$, $1 \leqslant k \leqslant N$, there are exactly*

$$\sum_{i=0}^{m-1} c_i q_i + (c_m - 1) q_m = N - q_m \text{ intervals of length } \frac{1}{q'_{m+1}} = \frac{a'_{m+2}}{q'_{m+2}}.$$

If $\sum_{i=0}^{m-1} c_i q_i \geqslant q_{m-1}$, *then there are in addition*

$$\sum_{i=0}^{m-1} c_i q_i - q_{m-1} \quad \text{intervals of length} \quad \frac{a'_{m+1} - c_m}{q'_{m+1}}$$

and

$$q_m + q_{m-1} - \sum_{i=0}^{m-1} c_i q_i \quad \text{intervals of length} \quad \frac{a'_{m+1} - c_m + 1}{q'_{m+1}}.$$

If, however, $\sum_{i=0}^{m-1} c_i q_i < q_{m-1}$, *then the additional intervals consist of*

$$\sum_{i=0}^{m-1} c_i q_i + q_m - q_{m-1} \quad \text{intervals of length} \quad \frac{a'_{m+1} - c_m + 1}{q'_{m+1}}$$

and

$$q_{m-1} - \sum_{i=0}^{m-1} c_i q_i \quad \text{intervals of length} \quad \frac{a'_{m+1} - c_m + 2}{q'_{m+1}}.$$

Proof. This corollary is deduced from theorem 1 by checking the lengths of the various subintervals of J_r. Clearly the points $P_r+(-1)^m s/q'_{m+1}$, $s=1,2,\dots,b$, divide the interval J_r into b intervals of length $1/q'_{m+1}$ and one interval of length $(a'_{m+1}-b)/q'_{m+1}$ if J_r is short or of length $(a'_{m+1}+1-b)/q'_{m+1}$ if J_r is long. The highest value b which occurs for s depends on $N-c_m q_m$. If $N-c_m q_m \geqslant q_{m-1}$ = number of long J_r then $b=c_m$ for all q_{m-1} long intervals and for $N-c_m q_m-q_{m-1}$ short intervals, whereas $b=c_m-1$ for the remaining $q_m-q_{m-1}-(N-c_m q_m-q_{m-1})$ short intervals. This gives the right number of intervals of the various lengths if $N-c_m q_m \geqslant q_{m-1}$. If $N-c_m q_m < q_{m-1}$ the counting argument is quite similar.

COROLLARY 2. *If m is even, then*

(2.11) $$\min_{1\leqslant k\leqslant N}\{k\xi\} = \{q_m\xi\} = \frac{1}{q'_{m+1}}$$

and

(2.12) $$\max_{1\leqslant k\leqslant N}\{k\xi\} = \begin{cases} \{(q_{m-1}+c_m q_m)\xi\} = 1-\dfrac{a'_{m+1}-c_m}{q'_{m+1}} & \text{if } q_{m-1}+c_m q_m \leqslant N, \\ \{(q_{m-1}+(c_m-1)q_m)\xi\} = 1-\dfrac{a'_{m+1}+1-c_m}{q'_{m+1}} & \text{if } q_{m-1}+c_m q_m > N. \end{cases}$$

If m is odd, then

(2.13) $$\min_{1\leqslant k\leqslant N}\{k\xi\} = \begin{cases} \{(q_{m-1}+c_m q_m)\xi\} = \dfrac{a'_{m+1}-c_m}{q'_{m+1}} & \text{if } q_{m-1}+c_m q_m \leqslant N, \\ \{(q_{m-1}+(c_m-1)q_m)\xi\} = \dfrac{a'_{m+1}+1-c_m}{q'_{m+1}} & \text{if } q_{m-1}+c_m q_m > N, \end{cases}$$

and

(2.14) $$\max_{1\leqslant k\leqslant N}\{k\xi\} = \{q_m\xi\} = 1-\frac{1}{q'_{m+1}}.$$

Moreover, (4)

(2.15) $$\min_{1\leqslant k\leqslant N}\|k\xi\| = \|q_m\xi\| = \frac{1}{q'_{m+1}}.$$

Proof. As an example we prove (2.12). The other formulae are proved in the same manner. For m even, $\lambda_{q_{m-1}} = q_{m-1}$ because of (2.6) and (2.7). Thus the point P_r in $((q_m-1)/q_m,\ q_m/q_m)$ equals $\{q_{m-1}\xi\}$ and $J_{q_{m-1}}$ is "long". The largest value of $\{k\xi\}$ is therefore achieved for $k = q_{m-1}+bq_m$ where b is the maximal value for which $q_{m-1}+bq_m \leqslant N$ (cf (2.10)). Again by (2.7), and (1.6),

$$\{(q_{m-1}+bq_m)\xi\} = 1-\frac{1}{q_m}+\frac{q_{m-1}+bq_m}{q_m q'_{m+1}} = 1-\frac{a'_{m+1}-b}{q'_{m+1}}.$$

This is indeed the value given in (2.12).

COROLLARY 3. *The maximal spacing $L_N(\xi)$ is given by*

(2.16) $$L_N(\xi) = 1-\max_{1\leqslant k\leqslant N}\{k\xi\}+\min_{1\leqslant k\leqslant N}\{k\xi\}.$$

In other words the maximal interval between adjacent points $\{k\xi\}$ is the interval containing 0 (*and* 1). (For a precise definition of L_N, see theorem 3 in the introduction.)

Proof. By corollary 1,

$$L_N(\xi) = \begin{cases} \dfrac{a'_{m+1}-c_m+1}{q'_{m+1}} & \text{if } N-c_m q_m \geqslant q_{m-1}, \\ \dfrac{a'_{m+1}-c_m+2}{q'_{m+1}} & \text{if } N-c_m q_m < q_{m-1}. \end{cases}$$

One immediately verifies from (2.11)-(2.14) that the value of $1-\max_{1\leqslant k\leqslant N}\{k\xi\} + \min_{1\leqslant k\leqslant N}\{k\xi\}$ always agrees with this.

We now quote a result of Surányi [13].

THEOREM (Surányi). *If ξ is irrational and $j_1/k_1 < \xi < j_2/k_2$ where j_1/k_1 and j_2/k_2 are successive members of F_N, then*

(2.17) $$\min_{1\leqslant k\leqslant N}\{k\xi\} = \{k_1\xi\} \quad \text{and} \quad \max_{1\leqslant k\leqslant N}\{k\xi\} = \{k_2\xi\}.$$

When we combine this with corollary 2 we obtain theorem 2 of the introduction.

We proceed with the proof of theorem 2. For $N=1$, the theorem is trivial and we may assume $N\geqslant 2$. For irrational ξ, $\min_{1\leqslant k\leqslant N}\{k\xi\}$ and $\max_{1\leqslant k\leqslant N}\{k\xi\}$ occur for unique values of k. Comparison of (2.11)-(2.14) with (2.17) shows that either

(i) m is even, $q_m = k_1$ and $q_{m-1}+\left[\dfrac{N-q_{m-1}}{q_m}\right]q_m = k_2$

or

(ii) m is odd, $q_m = k_2$ and $q_{m-1}+\left[\dfrac{N-q_{m-1}}{q_m}\right]q_m = k_1$.

If case (i) prevails, then

$$0 \leqslant q_{m-1} = k_2 - \text{integral multiple of } k_1 < q_m = k_1$$

and therefore

$$q_{m-1} = k_2 - \left[\frac{k_2}{k_1}\right] k_1$$

and in case (ii) the same argument with k_1 and k_2 interchanged is valid. (Only for (1.12) we have to rule out $q_{m-1} = q_m$, which can occur only if $q_m = q_{m-1} = 1$, $m = 1$, $q_m = k_2$.)

Since $j_1 k_1^{-1}$ and $j_2 k_2^{-1}$ are consecutive elements of F_N with $N \geqslant 2$, one has ([5], Chapter 3) $k_1 \neq k_2$ and

$$(2.18) \qquad j_2 k_1 - j_1 k_2 = 1 \quad \text{and} \quad \frac{j_2}{k_2} - \frac{j_1}{k_1} = \frac{1}{k_1 k_2}.$$

Thus

$$(2.19) \qquad 0 < \xi - \frac{j_1}{k_1} < \frac{1}{k_1 k_2} \leqslant \min \left| \xi - \frac{j}{k_1} \right|$$

where the minimum is over all $jk_1^{-1} \in F_N$ with $j \neq j_1$. The last inequality is obvious from the first two inequalities in (2.19) if $k_2 \geqslant 2$. But $k_2 = 1$ can occur only for $j_2 k_2^{-1} = 1$ and then for $j \leqslant j_1 - 1$ (2.19) is again obvious, whereas $jk_1^{-1} > j_2 k_2^{-1} = 1$ is impossible. Since by (2.1)

$$(2.20) \qquad \left| \xi - \frac{p_m}{q_m} \right| = \frac{1}{q_m q'_{m+1}} < \frac{1}{N q_m} \leqslant \frac{1}{2 q_m},$$

we conclude from (2.19) that in case (i) p_m must be j_1 and then, again by (2.1), (1.10) must hold. A similar argument is valid in case (ii) and it is only necessary to check which of the alternatives (i) or (ii) prevails for a given ξ. For this we refer to (2.15) and (2.20) which show that in case (i) one must have

$$\|q_m \xi\| = k_1 \left(\xi - \frac{j_1}{k_1} \right) < |1 - \{k_2 \xi\}| = k_2 \left(\frac{j_2}{k_2} - \xi \right)$$

or equivalently

$$\xi < \frac{j_1 + j_2}{k_1 + k_2}.$$

In case (ii) the inequalities have to be reversed. This completes the proof of theorem 2.

3. The distribution of the maximal spacing between points $\{k\xi\}$.

We give here the

Proof of theorem 3. Put

$$W(N, x) = \{\xi : N L_N(\xi) \leqslant x\}.$$

If

$$(3.1) \qquad \frac{j_1}{k_1} < \xi < \frac{j_2}{k_2}$$

where $j_1 k_1^{-1}$ and $j_2 k_2^{-1}$ are successive members of F_N (and hence $k_1 \neq k_2$ if $N \geqslant 2$ by theorem 31 of [5]), then by (2.18) and (2.19)

$$\{k_1 \xi\} = k_1 \left(\xi - \frac{j_1}{k_1} \right)$$

and

$$1 - \{k_2 \xi\} = k_2 \left(\frac{j_2}{k_2} - \xi \right) = \frac{1}{k_1} - k_2 \left(\xi - \frac{j_1}{k_1} \right).$$

Therefore, by corollary 3 and Surányi's theorem,

$$L_N(\xi) = k_1 \left(\xi - \frac{j_1}{k_1} \right) + k_2 \left(\frac{j_2}{k_2} - \xi \right)$$

whenever (3.1) holds. Using (2.18) once more, one has

$$(3.2) \qquad \left| W(N, x) \cap \left(\frac{j_1}{k_1}, \frac{j_2}{k_2} \right) \right| = \begin{cases} g(k_1, k_2, x, N) & \text{if} \quad k_1 > k_2, \\ g(k_2, k_1, x, N) & \text{if} \quad k_2 > k_1, \end{cases}$$

where

$$(3.3) \qquad g(k_1, k_2, x, N) = \min \left(\frac{1}{k_1 - k_2} \left(\frac{x}{N} - \frac{1}{k_1} \right)^+, \frac{1}{k_1 k_2} \right)$$

(c^+ stands for $\max(0, c)$). Consequently, for $N \geqslant 2$,

$$(3.4) \qquad |W(N, x)| = \sum_{1 \leqslant k_2 < k_1 \leqslant N} \sum_{j_1, j_2} g(k_1, k_2, x, N) + \sum_{1 \leqslant k_1 < k_2 \leqslant N} \sum_{j_1, j_2} g(k_2, k_1, x, N).$$

where the sum over j_1, j_2 is over those pairs j_1, j_2, for which $j_1 k_1^{-1} < j_2 k_2^{-1}$ are consecutive elements of F_N. It was proved by Friedman and Niven [4] (see also [3]) that there exists exactly one such pair j_1, j_2 if

$$(3.5) \qquad (k_1, k_2) = 1 \quad \text{and} \quad k_1 + k_2 > N.$$

Otherwise there is no such pair. Thus

$$|W(N, x)| = 2 \sum_{k_2 = 1}^{N} \sideset{}{'}\sum_{N - k_2 < k_1 \leqslant k_2} g(k_2, k_1, x, N)$$

where Σ' is only over those k_1 with $(k_1, k_2) = 1$. When (3.3) is substituted, this becomes

$$|W(N, x)| = 2 \sum_{\max\left(\frac{N}{x}, \frac{N}{2}\right) < k_2 \leqslant N} \left(\frac{x}{N} - \frac{1}{k_2}\right) \sum_{N-k_2 < k_1 \leqslant \frac{N}{x}}{}' \frac{1}{k_2 - k_1}$$
$$+ 2 \sum_{\max\left(\frac{N}{x}, \frac{N}{2}\right) < k_2 \leqslant N} \frac{1}{k_2} \sum_{\max\left(N-k_2, \frac{N}{x}\right) < k_1 \leqslant k_2}{}' \frac{1}{k_1}.$$

For $x < 1$ the sums are empty and $|W(N, x)| = 0$. For $1 \leqslant x \leqslant 2$ we obtain by means of lemma 2 of [8]

$$(3.6)\qquad |W(N, x)| = 2 \sum_{\frac{N}{x} < k_2 \leqslant N} \left(\frac{x}{N} - \frac{1}{k_2}\right) \frac{\Phi(k_2)}{k_2} \log \frac{2k_2 - N}{k_2 - Nx^{-1}}$$
$$+ 2 \sum_{\frac{N}{x} < k_2 \leqslant N} \frac{1}{k_2} \cdot \frac{\Phi(k_2)}{k_2} \log \frac{k_2}{Nx^{-1}} + O\left(\sum_{\frac{N}{x} < k_2 \leqslant N} \frac{d(k_2)}{k_2 N}\right).$$

Here, as in [8], $\Phi(\cdot)$ is Euler's function and $d(k_2) =$ number of divisors of k_2. Just as in the proof of theorem 1 of [8] the error term in (3.6) tends to zero as $N \to \infty$ and $\Phi(k_2)/k_2$ in the sums in (3.6) may be replaced by its "average value" $6/\pi^2$. One therefore obtains

$$|W(N, x)| = \frac{12}{\pi^2} \int_{x^{-1}}^{1} \left(x - \frac{1}{t}\right) \log \frac{2t-1}{t - x^{-1}}\, dt + \frac{12}{\pi^2} \int_{x^{-1}}^{1} \frac{1}{t} \log xt\, dt + o(1) \quad (N \to \infty).$$

The last case, where $x > 2$, is treated in a similar manner.

4. Criterion for boundedness of $R(M, \xi, a, b)$**.** This section is devoted to the proof of theorem 4. The fact that

$$(4.1)\qquad b - a = \{k\xi\}$$

implies

$$(4.2)\qquad |R(M, \xi, a, b)| \leqslant C(k)$$

for some constant C and all $M \geqslant 0$ was proved by Hecke [6] and Ostrowski [10]. (The precise value of $C(k)$ is not important here. Ostrowski gives $C(k) = |k|$ but this can be improved for most ξ's.) We therefore only have to prove that (4.1) is a necessary condition for (4.2). Except for a slight modification this was conjectured by Erdös and Szüsz ([2], p. 61). For ξ rational it is not difficult to see that boundedness of $R(M, \xi, a, b)$ implies that $b = \{k\xi\}$ for some ξ. In the sequel ξ will therefore be assumed to be a fixed irrational number. By a result of Bohl ([1], p. 226) the boundedness in M of $R(M, \xi, a, b)$ for a given ξ depends only on $b - a$ and not on a and b separately. It therefore suffices to take $a = 0$ and $0 < b < 1$ and for shortness we write $R(M, b)$ for $R(M, \xi, 0, b)$. We want to approximate b by points of the form $\{k\xi\}$, in particular we shall want a good approximation of this form with $k \leqslant q_n = q_n(\xi)$ for each n. For this purpose we apply theorem 1 with $N = q_n$. In this case $m(N, \xi) = n$ and theorem 1 states that exactly one point $\{k\xi\}$ with $k \leqslant q_n$ belongs to $(rq_n^{-1}, (r+1)q_n^{-1})$. This point was denoted by P_r if n is even and by $P_{q_n - r - 1}$ if n is odd. In agreement with (2.5) λ_r denotes the unique positive integer not exceeding q_n for which

$$(4.3)\qquad P_r = \{\lambda_r \xi\}.$$

It will be necessary in this section to indicate that P_r and λ_r depend on n. Accordingly we shall denote them by $P_r^{(n)}$ and $\lambda_r^{(n)}$. Similarly we shall write $J_r^{(n)}$ for the interval J_r introduced in theorem 1. For each n, there is a unique r_n such that [8]

$$(4.4)\qquad b \in J_{r_n}^{(n)} = \begin{cases} [P_{r_n}^{(n)}, P_{r_n+1}^{(n)}) & \text{if } n \text{ is even,} \\ (P_{r_n+1}^{(n)}, P_{r_n}^{(n)}] & \text{if } n \text{ is odd.} \end{cases}$$

To avoid cumbersone notation we shall use the following abbreviations:

$$(4.5)\qquad J(n) = J_{r_n}^{(n)}, \quad P(n) = P_{r_n}^{(n)}, \quad \lambda(n) = \lambda_{r_n}^{(n)}.$$

We now consider the multiples $\{(\lambda(n) + dq_n)\xi\}$ for which $\lambda(n) + dq_n \leqslant q_{n+1}$, $d = 0, 1, \ldots$ We always have, by the definition of $\lambda(n)$

$$(4.6)\qquad \lambda(n) \leqslant q_n.$$

If $\lambda(n) \leqslant q_{n-1}$ then the values $d = 0, 1, \ldots, a_{n+1}$ are permissable and $J(n)$ is a "long interval" (see theorem 1). If $q_{n-1} < \lambda(n) \leqslant q_n$ only the values $d = 0, 1, \ldots, a_{n+1} - 1$ are permissable and $J(n)$ is a "short interval". We put for n even,

$$(4.7)\qquad d_n = \textit{largest permissable } d \textit{ for which } \{(\lambda(n) + dq_n)\xi\} \leqslant b.$$

For odd n, we define d_n in the same way except for a reversal of the inequality in (4.7). To fix attention assume that n is even. One merely has to reverse most of the inequalities below to treat an odd n. We also assume

(8) This argument is reminiscent of theorem 1 in [11] part II.

that $0 \notin J(n)$. Since $0 < b < 1$, this holds for all sufficiently large n. Under these circumstances we have, by (4.3) and (2.10)

$$(4.8) \qquad \{(\lambda(n)+dq_n)\xi\} = P(n)+\frac{d}{q'_{n+1}},$$

and the definition of d_n therefore implies

$$(4.9) \qquad \{(\lambda(n)+d_n q_n)\xi\} = P(n)+\frac{d_n}{q'_{n+1}} \leqslant b < P(n)+\frac{d_n+1}{q'_{n+1}} = \{(\lambda(n)+(d_n+1)q_n)\xi\}$$

whenever d_n+1 is still a permissable value, i.e. if $\lambda(n)+(d_n+1)q_n \leqslant q_{n+1}$. This is certainly the case if

$$(4.10) \qquad d_n \leqslant a_{n+1}-2$$

which we shall assume for the time being. From now on we also assume that b is not of the form $\{k\xi\}$ for some integer k. The inequalities in (4.9) are then strict. Following an idea of Ostrowski [9], we shall now construct a sequence of M's, defined in terms of d_n and q_n for which $R(M, b)$ is unbounded. To begin with we take

$$(4.11) \qquad M_n = (d_n+1)q_n,$$

which is less than q_{n+1} because of (4.10), and estimate $R(M_n, b)$. Since $b \in J(n) = J^{(n)}_{r_n}$ and n even, one has

$$0 < P^{(n)}_0 < P^{(n)}_1 < \ldots < P^{(n)}_{r_n} < b < P^{(n)}_{r_n+1} < \ldots < P^{(n)}_{q_n-1}.$$

Consequently

$$(4.12a) \qquad J^{(n)}_i \subseteq [0, b) \qquad \text{if} \qquad i < r_n,$$

$$(4.12b) \qquad J^{(n)}_i \cap [0, b) = \emptyset \qquad \text{if} \qquad r_n < i \leqslant q_n-2,$$

$$(4.12c) \qquad J^{(n)}_{q_n-1} \cap [0, b) = [0, P_0) = \left[0, \frac{1}{q'_{n+1}}\right).$$

Among the $d_n q_n$ multiples $\{k\xi\}$, $q_n+1 \leqslant k \leqslant (d_n+1)q_n$, there are by theorem 1 exactly d_n in each interval $J^{(n)}_i$. Therefore for each $0 \leqslant i < r_n$ exactly the (d_n+1) points $\{k\xi\}$ with $1 \leqslant k \leqslant (d_n+1)q_n = M_n$ which belong to $J^{(n)}_i$ also belong to $[0, b)$, namely the points

$$P^{(n)}_i+\{dq_n\xi\} = P^{(n)}_i+\frac{d}{q'_{n+1}}, \qquad 0 \leqslant d \leqslant d_n.$$

This is still true for $i = r(n)$ because of (4.9). For $i > r_n$ no point in $J^{(n)}_i$ belongs to $[0, b)$. This is obvious for $r_n < i \leqslant q_n-2$ from (4.12b). For $i = q_n-1$ it follows from (2.10) with q_n-1 substituted for r. These data prove

$$(4.13) \qquad N(M_n, \xi, 0, b) = (d_n+1)(r_n+1).$$

On the other hand, by (4.3) and (2.6)

$$(4.14) \qquad P(n) = \{\lambda(n)\xi\} = \left\{\frac{\lambda(n)p_n}{q_n}+\frac{\lambda(n)}{q_n q'_{n+1}}\right\} = \frac{r_n}{q_n}+\frac{\lambda(n)}{q_n q'_{n+1}}$$

and (4.9) and (4.6) therefore imply

$$(4.15) \qquad b \leqslant \frac{r_n}{q_n}+\frac{\lambda(n)}{q_n q'_{n+1}}+\frac{d_n+1}{q'_{n+1}} \leqslant \frac{r_n}{q_n}+\frac{d_n+2}{q'_{n+1}}.$$

Combining this with (4.13) and (1.6) we obtain

$$(4.16) \qquad R(M_n, b) = N(M_n, \xi, 0, b)-M_n b \geqslant \frac{d_n+1}{q'_{n+1}}\left((a'_{n+1}-d_n-2)q_n+q_{n-1}\right) \geqslant \frac{d_n+1}{a_{n+1}+2}\left(a_{n+1}-d_n-2+\frac{1}{a_{n+2}+1}\right).$$

It is easy to conclude from this

$$(4.17) \qquad R(M_n, b) \geqslant \frac{1}{28},$$

whenever

$$(4.18a) \qquad 0 \leqslant d_n \leqslant a_{n+1}-3$$

or

$$(4.18b) \qquad 0 \leqslant d_n = a_{n+1}-2 \qquad \text{and} \qquad a_{n+2} \leqslant 6.$$

Because of the assumption that $b \neq \{k\xi\}$ for all k there exists an $\varepsilon_n > 0$ such that the number of $1 \leqslant k \leqslant M_n$ with $0 < \{\varepsilon+k\xi\} < b = N(M_n, \xi, 0, b)$ whenever $|\varepsilon| < \varepsilon_n$.

In particular this holds for

$$\varepsilon = \left\{\sum_{j \geqslant n+s} e_j q_j \xi\right\}$$

whenever e_j integral, $|e_j| \leqslant a_{j+1}$ and s sufficiently large, say $s \geqslant s_n$. In fact,

$$\left\{\sum_{j \geqslant n+s} e_j q_j \xi\right\} \leqslant \sum_{j \geqslant n+s} |e_j| \{q_j \xi\} \leqslant \sum_{j \geqslant n+s} \frac{a_{j+1}}{q_{j+1}} \leqslant \sum_{j \geqslant n+s} \frac{1}{q_j} \leqslant \frac{4}{q_{n+s}} \leqslant 2^{3-(n+s)/2}$$

since

$$q_{j+2} \geqslant 2q_j.$$

We therefore obtain for $|e_j| \leqslant a_{j+1}$, $s \geqslant s_n$

(4.19) $$N\Big(\sum_{j\geqslant n+s} e_j q_j + M_n, \xi, 0, b\Big) - N\Big(\sum_{j\geqslant n+s} e_j q_j, \xi, 0, b\Big)$$
$$= \text{number of } 1 \leqslant k \leqslant M_n \text{ with } 0 \leqslant \Big\{\sum_{j\geqslant n+s} e_j q_j \xi + k\xi\Big\} < b$$
$$= N(M_n, \xi, 0, b).$$

Assume now that for infinitely many even n (4.18a) or (4.18b) holds. We can then select a subsequence $\{n_i\}$ for which (4.18a) or (4.18b) holds and such that

$$n_i + s_{n_i} \leqslant n_{i+1}.$$

By (4.19) we have then for

$$M = \sum_{i=1}^{t} M_{n_i} = \sum_{i=1}^{t} (d_{n_i}+1) q_{n_i},$$

(4.20) $$N(M, \xi, 0, b) = \sum_{j=1}^{t} \Big(N\Big(\sum_{i=j+1}^{t} M_{n_i} + M_{n_j}, \xi, 0, b\Big)$$
$$- N\Big(\sum_{i=i+1}^{t} M_{n_i}, \xi, 0, b\Big)\Big) = \sum_{j=1}^{t} N(M_{n_j}, \xi, 0, b)$$

and, by (4.17)

$$R(M, b) = \sum_{j=1}^{t} R(M_{n_j}, b) \geqslant \frac{t}{28}.$$

Since t can be taken arbitrary large, we see that R is unbounded if (4.18) holds for infinitely many even n. The same conclusion is valid if (4.18) holds for infinitely many odd n. From now on we may assume therefore that for $n \geqslant n_0$

(4.21a) $$0 \leqslant a_{n+1} - 1 \leqslant d_n \leqslant a_{n+1}$$

(since $d_n \leqslant a_{n+1}$ by definition), or

(4.21b) $$0 \leqslant d_n = a_{n+1} - 2 \quad \text{and} \quad a_{n+2} \geqslant 7.$$

We now investigate closer what happens if (4.21b) holds for infinitely many n. For the sake of argument assume again that $n \geqslant n_0$ is even and that (4.21b) holds. (4.9) (with strict inequalities) states

(4.22) $$\{(\lambda(n) + d_n q_n)\xi\} < b < \{(\lambda(n) + (d_n+1) q_n)\xi\}.$$

But

$$\lambda(n) + d_n q_n < \lambda(n) + (d_n+1) q_n \leqslant a_{n+1} q_n < q_{n+1}.$$

Moreover, by theorem 1, there is no $k \leqslant q_{n+1}$ for which

$$\{(\lambda(n) + d_n q_n)\xi\} < \{k\xi\} < \{(\lambda(n) + (d_n+1) q_n)\xi\}.$$

In other words,

$$P' = \{(\lambda(n) + d_n q_n)\xi\} \quad \text{and} \quad P'' = \{(\lambda(n) + (d_n+1) q_n)\xi\}$$

are two adjacent points among the $P_r^{(n+1)}$. Thus according to (4.4), (4.5), and (4.22) we must have ($n+1$ is odd)

(4.23) $$J(n+1) = (P', P''],$$
$$P(n+1) = P'' = \{(\lambda(n) + (d_n+1) q_n)\xi\},$$
$$\lambda(n+1) = \lambda(n) + (d_n+1) q_n.$$

The analogue of one half of (4.9) at the $(n+1)$st stage becomes (recall that $(n+1)$ is odd)

$$b < \{(\lambda(n+1) + d_{n+1} q_{n+1})\xi\} = P(n+1) - \frac{d_{n+1}}{q'_{n+2}} = P(n) + \frac{d_n+1}{q'_{n+1}} - \frac{d_{n+1}}{q'_{n+2}}.$$

If we now substitute $d_n = a_{n+1} - 2$ and use the fact that $d_{n+1} \geqslant a_{n+2} - 2$ since $n+1 \geqslant n \geqslant n_0$, we obtain in the same manner as in (4.15)

$$b < \frac{r_n}{q_n} + \frac{a_{n+1} - (a_{n+2} - 2)/a'_{n+2}}{q'_{n+1}}.$$

With

$$M_n = (a_{n+1} - 1) q_n$$

as in (4.11), (4.13) remains valid and (4.16) can now be sharpened to

$$R(M_n, b) \geqslant \frac{a_{n+1} - 1}{a_{n+1} + 2} \cdot \frac{a_{n+2} - 2}{a'_{n+2}} \geqslant \frac{1}{4} \cdot \frac{5}{8}.$$

since $a_{n+1} = d_n + 2 \geqslant 2$ and $a_{n+2} \geqslant 7$. As before we derive from this that $R(M, b)$ is unbounded if (4.21b) occurs infinitely often. Thus if R is bounded we may assume that (4.21a) holds as soon as n exceeds a certain n_1. We proceed to limit the possibilities for d_n still further. Assume that $n > n_1$ and that

(4.24a) $$d_n = a_{n+1}$$

or

(4.24b) $$d_n = a_{n+1} - 1 \quad \text{and} \quad J(n) \text{ is "short" (i.e. } \lambda(n) > q_{n-1}).$$

(assumption (4.10) is dropped now). In both cases d_n has the maximal permissable value of d for which $\lambda(n) + d q_n \leqslant q_{n+1}$. Let n be even again. (4.9) now has to be replaced by

(4.25) $$\{(\lambda(n) + d_n q_n)\xi\} < b < P_{r_n+1}^{(n)} = \{\lambda_{r_n+1}^{(n)}\xi\}$$

since $P_{r_n+1}^{(n)}$ is the right-hand end point of $J(n)$ and there is no $k \leqslant q_{n+1}$ with

$$\{(\lambda(n) + d_n q_n)\xi\} < \{k\xi\} < \{\lambda_{r_n+1}^{(n)}\xi\}.$$

The argument which led from (4.22) to (4.23) now shows that

(4.26a) $$J(n+1) = \left[\{(\lambda(n)+d_n q_n)\xi\},\ \{\lambda^{(n)}_{r_n+1}\xi\}\right],$$

(4.26b) $$P(n+1) = P^{(n)}_{r_n+1} = \{\lambda^{(n)}_{r_n+1}\xi\},$$

(4.26c) $$\lambda(n+1) = \lambda^{(n)}_{r_n+1} \leqslant q_n.$$

The last inequality follows from the definition of $\lambda^{(n)}_r$ (see (4.3)) and will be crucial for our argument. In particular it implies that $J(n+1)$ is a "long interval" and $\lambda(n+1)+a_{n+2}q_{n+1} \leqslant q_{n+2}$. Since $n > n_1$, d_{n+1} can only take the values $a_{n+2}-1$ and a_{n+2}.

If we assume

(4.27) $$d_{n+1} = a_{n+2}-1,$$

the analogue of (4.9) at the $(n+1)$st stage is

(4.28) $$P(n+1)+\{(d_{n+1}+1)q_{n+1}\xi\} = P(n+1)-\frac{a_{n+2}}{q'_{n+2}} < b$$
$$< P(n+1)+\{d_{n+1}q_{n+1}\xi\} = P(n+1)-\frac{a_{n+2}-1}{q'_{n+2}},$$

since $d_{n+1}+1 = a_{n+2}$ is a permissable value for d and $n+1$ is odd. In turn this implies

$$P(n+2) = P(n+1)+\{a_{n+2}q_{n+1}\xi\}$$

and finally

(4.29) $$b > P(n+2)+\{d_{n+2}q_{n+2}\xi\} \geqslant P(n+1)-\frac{a_{n+2}}{q'_{n+2}}+\frac{a_{n+3}-1}{q'_{n+3}},$$

since $d_{n+2} \geqslant a_{n+3}-1$ for $n+2 \geqslant n > n_1$.

Because (compare (4.14))

$$P(n+1) = P^{(n+1)}_{r_{n+1}} = \{\lambda(n+1)\xi\} = \frac{q_{n+1}-r_{n+1}}{q_{n+1}} - \frac{\lambda(n+1)}{q_{n+1}q'_{n+2}}$$

we obtain from (4.29) and (4.26c)

(4.30) $$b > \frac{q_{n+1}-r_{n+1}}{q_{n+1}} - \frac{q_n+a_{n+2}q_{n+1}}{q_{n+1}q'_{n+2}} + \frac{a_{n+3}-1}{q'_{n+3}}$$
$$= \frac{q_{n+1}-r_{n+1}}{q_{n+1}} - \frac{q_{n+2}}{q_{n+1}q'_{n+2}} + \frac{a_{n+3}-1}{a'_{n+3}}\cdot\frac{1}{q'_{n+2}}.$$

Under these circumstances we choose

$$M_{n+1} = a_{n+2}q_{n+1},$$

and claim that

(4.31) $$N(M_{n+1}, \xi, 0, b) = a_{n+2}(q_{n+1}-r_{n+1}-1).$$

Indeed none of the points $P(n+1)+\{cq_{n+1}\xi\}$, $c \leqslant a_{n+2}-1$, will belong to $[0, b)$ by the second inequality of (4.28). In each interval $J^{(n+1)}_r$ with $r_{n+1} < r \leqslant q_{n+1}-1$ there will be exactly a_{n+2} points $\{k\xi\}$, $k \leqslant M_{n+1}$, by theorem 1, and all of them belong to $[0, b)$ and none of the points $\{k\xi\}$ in $J^{(n+1)}_r$ with $r < r_{n+1}$ belong to $[0, b)$. (This argument is merely a repetition of the proof of (4.13), now with an odd index). From (4.30) and (4.31) we conclude

$$R(M_{n+1}, b) \leqslant -\frac{a_{n+2}\cdot a_{n+3}q_{n+1}}{a'_{n+3}\cdot q'_{n+2}} \leqslant -\frac{a_{n+2}}{a_{n+2}+2}\cdot\frac{a_{n+3}}{a_{n+3}+1} \leqslant -\frac{1}{6}.$$

As before this can only happen a finite number of times if $R(M, b)$ is to remain bounded and therefore (4.24) and (4.27) together can only happen a finite number of times. Thus if R remains bounded we may assume that for every $n \geqslant n_2$

$$a_{n+1}-1 \leqslant d_n \leqslant a_{n+1}$$

but both (4.24a) and (4.24b) fail or (4.27) fails. This only leaves the following possibilities for d_n, $n \geqslant n_2$.

(i) $d_n = a_{n+1}$. Then (4.27) must fail and hence $d_{n+1} = a_{n+2}$ and then $d_{n+i} = a_{n+i+1}$ for $i \geqslant 0$.

(ii) $d_n = a_{n+1}-1$ *and* $J(n)$ *is a "short interval"*. Again (4.27) must fail, hence $d_{n+1} = a_{n+2}$ and then by case (i) $d_{n+i} = a_{n+i+1}$ for $i \geqslant 1$.

(iii) $d_n = a_{n+1}-1$ *and* $J(n)$ *is a "long interval"*. Then $\lambda(n)+a_{n+1}q_n \leqslant q_{n+1}$ and (4.9) is still valid. By the argument leading from (4.22) to (4.23) we conclude that

$$J(n+1) = \left(\{(\lambda(n)+d_nq_n)\xi\},\ \{(\lambda(n)+(d_n+1)q_n)\xi\}\right)$$

which has length

$$\{q_n\xi\} = \frac{1}{q'_{n+1}} = \frac{a'_{n+2}}{q'_{n+2}}$$

and is therefore a short $J^{(n+1)}_r$. At the $(n+1)$st step we are therefore in case (i) or case (ii) and $d_{n+i} = a_{n+i+1}$ for $i \geqslant 2$.

The final conclusion is that if b is not of the form $\{k\xi\}$, then $R(M, b)$ can only be bounded if $d_n = a_{n+1}$ for $n \geqslant n_3 = n_2+2$. However, as remarked before (4.7), $d_n = a_{n+1}$ can occur only if $J(n)$ is a long interval and in addition it was proved in (4.26) that $d_n = a_{n+1}$ implies

$$P(n+1) = P^{(n)}_{r_n+1} = P^{(n)}_{r_n}+(-1)^n\cdot\text{length of } J_n$$
$$= P(n)+(-1)^n\frac{a'_{n+1}+1}{q'_{n+1}} = P(n)+\{q_n\xi\}-\{q_{n-1}\xi\}+(-1)^n.$$

Iteration of this formula shows

$$P(n) = P(n_3) + \sum_{j=n_3}^{n-1} (\{q_j \xi\} - \{q_{j-1} \xi\}) + \tfrac{1}{2}(-1)^{n_3} - \tfrac{1}{2}(-1)^n$$
$$= \{\lambda(n_3)\xi\} + \{q_{n-1}\xi\} - \{q_{n_3-1}\xi\} + \tfrac{1}{2}(-1)^{n_3} - \tfrac{1}{2}(-1)^n$$

and therefore (see (4.4))

$$b = \lim_{n\to\infty} P(n) = \{\lambda(n_3)\xi\} - \{q_{n_3-1}\xi\} + \tfrac{1}{2}(1+(-1)^{n_3}) = \{(\lambda(n_3) - q_{n_3-1})\xi\}$$

which is after all of the form $\{k\xi\}$. Thus $R(M, b)$ cannot be bounded unless (4.1) holds.

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